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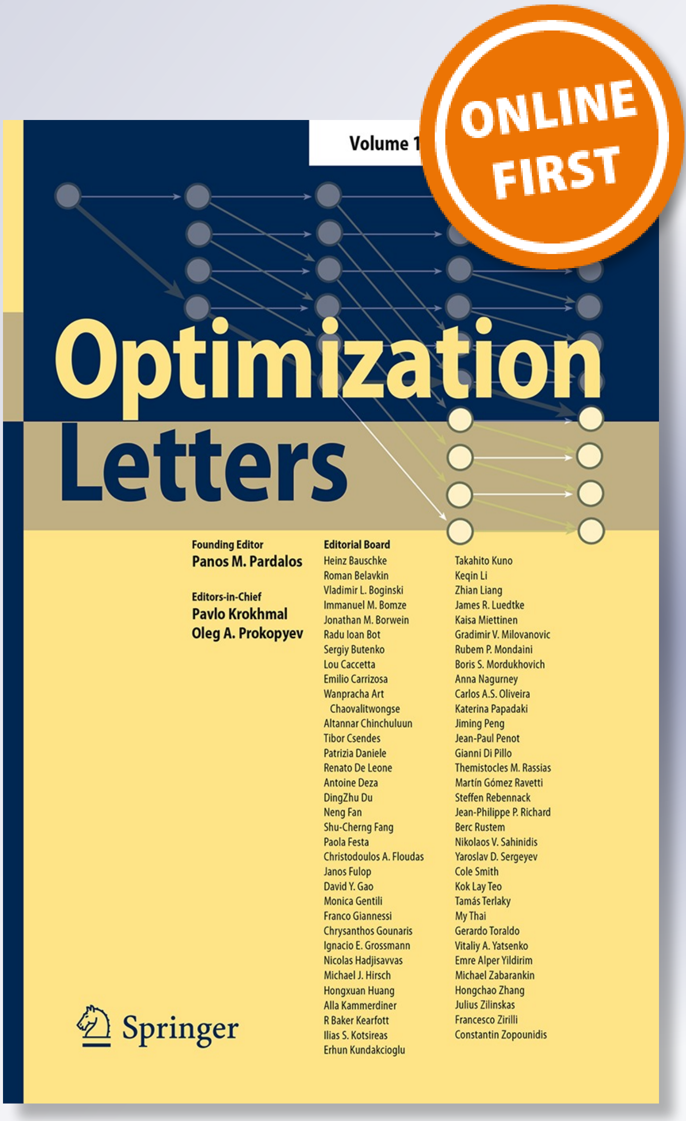

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ORIGINAL PAPER

General results on preferential attachment and clustering coefficient

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Abstract This is a review paper that covers some recent results on the behavior of the clustering coefficient in preferential attachment networks and scale-free networks in general. The paper focuses on *general approaches* to network science. In other words, instead of discussing different fully specified random graph models, we describe some generic results which hold for classes of models. Namely, we first discuss a generalized class of preferential attachment models which includes many classical models. It turns out that some properties can be analyzed for the whole class without specifying the model. Such properties are the degree distribution and the global and average local clustering coefficients. Finally, we discuss some surprising results on the behavior of the global clustering coefficient in scale-free networks. Here we do not assume any underlying model.

Keywords Networks · Random graph models · Preferential attachment · Power-law degree distribution · Clustering coefficient

1 Introduction

Many social, biological, and information systems can be represented by networks, whose vertices are items and links are relations between these items [1,2,7,10]. That is why the evolution of complex networks attracted a lot of attention in recent years [16,33]. In particular, numerous random graph models have been proposed to reflect and predict important quantitative and topological aspects of growing real-world networks. Such models are used in many fields: experimental physics, bioinformatics,

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information retrieval, data mining, etc. An extensive review can be found elsewhere (e.g., see [1,7,8]).

It turns out that many real-world networks of diverse nature have some typical properties: small diameter, power-law degree distribution, high clustering, and others [22,31–33,38,43]. Probably the most extensively studied property of networks is their vertex degree distribution. For the majority of studied real-world networks, the portion of vertices of degree d was observed to decrease as $d^{-\gamma-1}$, usually with $1 < \gamma < 2$ [3–5,7,13,21]. Such networks are often called scale-free. Sometimes, the *cumulative* degree distribution is considered: the portion of vertices of degree greater than d decreases as $d^{-\gamma}$. In this paper, the parameter γ corresponds to the slope of cumulative degree distribution.

Another important characteristic of a network is its clustering coefficient, a measure capturing the tendency of a network to form clusters, densely interconnected sets of vertices. Several definitions of the clustering coefficient can be found in the literature [8]. In this paper, we consider the most popular two: the *global clustering coefficient* and the *average local clustering coefficient*. The global clustering coefficient $C_1(G)$ is the ratio of three times the number of triangles to the number of pairs of adjacent edges in G. The average local clustering coefficient is defined as follows: $C_2(G) = \frac{1}{n}\sum_{i=1}^{n} C(i)$, where $C(i) = \frac{T^i}{P_2^i}$ is the local clustering coefficient for a vertex i, T^i is the number of edges between the neighbors of the vertex i, and P_2^i is the number of pairs of neighbors. It is believed that for many real-world networks both the average local and the global clustering coefficients tend to a non-zero limit as the networks become large. Indeed, in many observed networks the values of both clustering coefficients are considerably high [33].

This paper reviews several recent works in which the above properties—the degree distribution and the clustering coefficient—are analyzed. The goal of the paper is to cover some general approaches to the analysis of these properties. We would like to remark that a huge number of different models have been proposed recently and covering all of these models is outside the scope of this paper. However, we do describe several classical models as an illustration to a generalized approach.

The most well-known approach to the modeling of complex networks is the *preferential attachment*. The main idea of this approach is that the graph is constructed step by step and at each time step a new vertex is added to the graph and is joined to m different vertices already existing in the graph chosen with probabilities proportional to their degrees. Many different models are based on the idea of preferential attachment: LCD [9], Buckley and Osthus [14], Holme and Kim [25], RAN [44], and many others. Preferential attachment is a natural process which allows to obtain a graph with a power-law degree distribution. We discuss the preferential attachment approach and several models based on this approach in Sect. 2.

Then, we present a general framework for analyzing preferential attachment models which was first proposed in [37]. The authors introduced a class of models (PA-class) defined in terms of constraints that are sufficient for the study of the degree distribution. The PA-class includes many classical preferential attachment models. We discuss this class and the degree distribution for the models of this class in Sect. 3.

In turned out that an additional constraint on the models in the PA-class allows to analyze the behavior of both average local and global clustering coefficients in this class (see Sect. 4). Moreover, if the parameter of the cumulative degree distribution γ belongs to $(1, 2)$, then the global clustering coefficient tends to zero for *any* model from the PA-class [37]. In Sect. 5, we present a result which further generalizes this observation [36]: namely, for *any* sequence of graphs with a power-law degree distributions with a parameter $\gamma \in (1, 2)$ the global clustering coefficient tends to zero. This result is quite surprising, since there is a common belief that for many real-world networks both the average local and the global clustering coefficients tend to a non-zero limit as the networks become large. In addition to the upper bound for the global clustering coefficient, in Sect. 5 we also present an algorithm which allows to construct a graph with nearly maximum (up to $n^{o(1)}$ multiplier) clustering coefficient given a power-law degree distribution.

Note that we are not aiming at providing the complete proofs for all the results covered in this paper. However, we usually describe the main ideas and tools. In some cases, if a proof is easy to follow, we also provide a sketch of the proof.

2 Preferential attachment models

In 1999, Barabási and Albert observed [3] that the degree distribution of the World Wide Web follows a power law with a parameter ~2.1 (or 1.1 for the cumulative distribution). As a possible explanation for this phenomenon, they proposed a graph construction stochastic process governed by the *preferential attachment*. At each time step of the process, a new vertex is added to the graph and is joined to m different vertices already existing in the graph chosen with probabilities proportional to their degrees.

Denote by d_v^n the degree of a vertex v in the growing graph at time n. At each step m edges are added, so we have $\sum_v d_v^n = 2mn$. This observation and the preferential attachment rule imply that

$$\mathsf{P}(d_v^{n+1} = d + 1 \mid d_v^n = d) = \frac{d}{2n}. \tag{1}$$

Note that the condition (1) on the attachment probability does not specify the distribution of m vertices to be joined to, in particular their dependence. Therefore, it would be more accurate to say that Barabási and Albert proposed not a single model, but a class of models. As it was shown later by Bollobás and Riordan, there is a whole range of models that fit the Barabási–Albert description, but possess very different behavior.

Theorem 1 (Bollobás and Riordan [8]) *Let $f(n)$, $n \geq 2$, be any integer valued function with $f(2) = 0$ and $f(n) \leq f(n+1) \leq f(n) + 1$ for every $n \geq 2$, such that $f(n) \to \infty$ as $n \to \infty$. Then there is a random graph process $T(n)$ satisfying (1) such that, with probability 1, $T(n)$ has exactly $f(n)$ triangles for all sufficiently large n.*

Further in this section, we review several classical models based on the idea of preferential attachment. In the next section, we present an approach which allows to generalize all these models.

2.1 LCD-model

In [9], Bollobás and Riordan proposed a precisely defined model of the Barabási–Albert type, known as the LCD-model. The graph is constructed according to the following procedure. Let n be a number of vertices in our graph and m be a fixed positive integer parameter. We begin with the case $m = 1$. We inductively construct a random graph G_1^n. Start with G_1^1—the graph with one vertex and one loop. Assume that we already constructed a graph G_1^{t-1}. At the next step we add one vertex t and one edge between vertices t and i, where i is chosen randomly with the following probability:

$$\mathsf{P}(i = s) = \begin{cases} d_s^{t-1}/(2t-1) & \text{if } 1 \le s \le t-1, \\ 1/(2t-1) & \text{if } s = t. \end{cases} \tag{2}$$

In other words, the probability that a new vertex will be connected to a vertex i is proportional to the current degree of i. To obtain G_m^n with $m > 1$, we first construct G_1^{mn}. Then we identify the vertices $1, \dots, m$ to form the first vertex; we identify the vertices $m + 1, \dots, 2m$ to form the second vertex; and so on. After this procedure, edges from G_1^n connect "big" vertices in G_m^n. According to this definition multiple edges and loops may occur.

In [9], Bollobás and Riordan proved that for $d < n^{\frac{1}{15}}$ the portion of vertices of degree d asymptotically almost surely obeys the power law with the parameter 3. In [23], Grechnikov improved this result by removing the restriction on d.

It was also shown that the expected global clustering coefficient in the LCD-model is asymptotically proportional to $\frac{(\log n)^2}{n}$ and therefore tends to zero as the graph grows [8].

2.2 Buckley–Osthus model

One obtains a natural generalization of the LCD-model, requiring the probability of attachment to a vertex v to be proportional to $d_v^n + m\beta$, where β is a constant representing the *initial attractiveness* of a vertex [17,18]. Buckley and Osthus [14] proposed a precisely defined model with a nonnegative integer β. Their model is similar to the LCD-model, but Eq. (2) should be replaced by

$$\mathsf{P}(i = s) = \begin{cases} \frac{d_s^{t-1}+\beta}{(\beta+2)t-1} & \text{if } 1 \le s \le t-1, \\ \frac{\beta+1}{(\beta+2)t-1} & \text{if } s = t. \end{cases}$$

Móri [29] proposed a similar model which generalizes the Buckley–Osthus model to real $\beta > -1$. For both models, the degree distribution was shown to follow the power law with the parameter $3 + \beta$ in the range of small degrees [14,24,30].

The result of Eggemann and Noble [19] implies that the expected global clustering coefficient in the Móri model with $\beta > 0$ is asymptotically proportional to $\frac{\log n}{n}$. For $\beta = 0$, the Móri model is almost identical to the LCD-model. Therefore, the authors of [19] emphasize the confusing difference between the clustering coefficients $\left(\frac{(\log n)^2}{n}\text{ versus }\frac{\log n}{n}\right)$.

2.3 Holme–Kim model

The main drawback of the described preferential attachment models is an unrealistic behavior of the clustering coefficient. In fact, for all discussed models the clustering coefficient tends to zero as a graph grows, while in many real-world networks the clustering coefficient is approximately a constant [7].

A model with an asymptotically constant average local clustering coefficient was proposed by Holme and Kim [25]. The idea is to mix preferential attachment steps with steps of triad formation. Namely, when a new vertex appears, we add m edges in m steps. There are two types of steps:

- Preferential attachment (PA): an edge is attached to an existing vertex with the probability proportional to its degree, as in the LCD-model.
- Triad formation (TF): if an edge between v and w was added in the previous PA step, then we add one more edge from v to a randomly chosen neighbor of w. If there remains no pair to connect, i.e., if all neighbors of w were already connected to v, do a PA step instead.

When a vertex v with m edges is added to the existing graph, we first perform one PA step, and then perform a TF step with the probability P_t or a PA step with the probability $1 - P_t$.

This model allows to tune the clustering coefficient by varying the probability of the triad formation step P_t. However, experiments and empirical analysis [25] show that the degree distribution in this model obeys the power law with the fixed parameter close to 3, which does not suit most real networks. In addition, as we demonstrate in Sect. 3, while the average local clustering coefficient for this model does not tend to zero as a graph grows, the global clustering coefficient still does.

2.4 RAN model

The random Apollonian network model (RAN) proposed in [44] is another interesting example of a Barabási–Albert type model with an asymptotically constant average local clustering coefficient. This model is based on a geometrical representation of a graph and it allows to construct a planar network. In order to construct a graph, one starts with a triangle (three vertices and three edges). Then, at each time step, a triangle (in the current graph drawn on a plane) is randomly selected, a new vertex is added inside this triangle and linked to the three vertices of this triangle.

This model allows to get a constant average local clustering coefficient and the degree distribution in this model obeys the power law with the parameter 3 [44].

There are many other models, not mentioned here, that are based on the idea of preferential attachment. For some of these models similar theorems on the degree distribution and the clustering coefficient are proved. The methods used in the proofs are also very similar. In the next section, we consider an approach [37] which provides a general framework for these models. Namely, we discuss a class of preferential attachment models that generalizes the models mentioned above, as well as many others.

3 Generalized preferential attachment

In this section, we present a generic approach to preferential attachment proposed in [37]. First, we formally define the PA-class of models; then, we discuss the degree distribution in these models.

3.1 Definition

Let us define a class of preferential attachment random graph models which generalizes several models discussed in the previous section.

Let G_m^n $(n \geq n_0)$ be a graph with n vertices $\{1, \dots, n\}$ and mn edges obtained as a result of the following process. We start at the time n_0 from an arbitrary graph $G_m^{n_0}$ with n_0 vertices and mn_0 edges. On the $(n+1)$-th step $(n \geq n_0)$, we make the graph G_m^{n+1} from G_m^n by adding a new vertex $n+1$ and m edges connecting this vertex to some m vertices from the set $\{1, \dots, n, n+1\}$. Recall that we denote by d_v^n the degree of a vertex v in G_m^n. If for some constants A and B the following conditions are satisfied

$$\mathbf{P}(d_v^{n+1} = d_v^n \mid G_m^n) = 1 - A\frac{d_v^n}{n} - B\frac{1}{n} + O\left(\frac{\left(d_v^n\right)^2}{n^2}\right), \quad 1 \leq v \leq n, \tag{3}$$

$$\mathbf{P}(d_v^{n+1} = d_v^n + 1 \mid G_m^n) = A\frac{d_v^n}{n} + B\frac{1}{n} + O\left(\frac{\left(d_v^n\right)^2}{n^2}\right), \quad 1 \leq v \leq n, \tag{4}$$

$$\mathbf{P}(d_v^{n+1} = d_v^n + j \mid G_m^n) = O\left(\frac{\left(d_v^n\right)^2}{n^2}\right), \quad 2 \leq j \leq m, \quad 1 \leq v \leq n, \tag{5}$$

$$\mathbf{P}(d_{n+1}^{n+1} = m + j) = O\left(\frac{1}{n}\right), \quad 1 \leq j \leq m, \tag{6}$$

then the random graph process G_m^n is a model from the PA-class. Here, as in [37], we require $2mA + B = m$ and $0 \leq A \leq 1$.

Note that even if we fix A and m, we still do not specify a concrete procedure for constructing a network, since we do not completely define the joint distribution of m endpoints of new edges. Therefore, there is a range of models possessing very different properties and satisfying the conditions (3)–(6). For example, the LCD, the Holme–Kim, and the RAN models belong to the PA-class with $A = 1/2$ and $B = 0$. The Buckley–Osthus (Móri) model also belongs to the PA-class with $A = \frac{1}{2+\beta}$ and $B = \frac{m\beta}{2+\beta}$.

It turns out, that some rigorous results can be proven for the whole PA-class without specifying a concrete model. For example, we can show a power-law degree distribution (Sect. 3.2). It is also possible to analyze both global and average local clustering coefficients (Sect. 4). Directions for future research include analyzing the maximum degree d_{max} and the average degree of the nearest neighbors of vertices with degree d [38], denoted as $k_{nn}(d)$, in the whole PA-class. It is also worth mentioning that by the definition all models from the PA-class are m-degenerate. Constant degeneracy implies, for example, that maximum clique problem can efficiently be solved for all such graphs [11].

3.2 Power-law degree distribution

Even though the distribution of vertices a new vertex is going to be connected to is not fully specified, it is still possible to analyze the degree distribution of all models in the PA-class.

By $N_n(d)$ denote the number of vertices of a given degree d in G_m^n. The following result on the expectation of $N_n(d)$ is proven in [37].

Theorem 2 (Ostroumova et al. [37]) *For every $d \geq m$ we have $\mathsf{E}N_n(d) = c(m,d)(n + O(d^{2+\frac{1}{A}}))$, where*

$$c(m,d) = \frac{\Gamma\left(d+\frac{B}{A}\right)\Gamma\left(m+\frac{B+1}{A}\right)}{A\Gamma\left(d+\frac{B+A+1}{A}\right)\Gamma\left(m+\frac{B}{A}\right)} \overset{d\to\infty}{\sim} \frac{\Gamma\left(m+\frac{B+1}{A}\right)d^{-1-\frac{1}{A}}}{A\Gamma\left(m+\frac{B}{A}\right)}$$

and $\Gamma(x)$ is the gamma function.

Theorem 2 can be proven by induction on d and n. Given a graph G_m^n, we can express the conditional expectation for the number of vertices of degree d in G_m^{n+1} (i.e., $\mathsf{E}(N_{n+1}(d) \mid G_m^n)$) in terms of $N_n(d)$, $N_n(d-1), \ldots, N_n(d-m)$. Here we only need the fact that the probability of having an edge between the vertex $n+1$ and a vertex v depends on the degree of v (see Eq. (3)). Using the law of total expectation, we can obtain the recurrent relation for $\mathsf{E}N_{n+1}(d)$ and prove the statement of Theorem 2 by induction.

It can also be shown that the number of vertices of a given degree d is highly concentrated around its expectation.

Theorem 3 (Ostroumova et al. [37]) *For every model from the PA-class and for every $d = d(n)$ we have*

$$\mathsf{P}\left(|N_n(d) - \mathsf{E}N_n(d)| \geq d\sqrt{n}\log n\right) = O(n^{-\log n}).$$

Therefore, for any $\delta > 0$ there exists a function $\varphi(n) = o(1)$ such that

$$\lim_{n\to\infty} \mathsf{P}(\exists\, d \leq n^{\frac{A-\delta}{4A+2}} : |N_n(d) - \mathsf{E}N_n(d)| \geq \varphi(n)\,\mathsf{E}N_n(d)) = 0.$$

The Azuma–Hoeffding inequality can be used to prove this theorem.

Theorem 4 (Azuma and Hoeffding) *Let* $(X_i)_{i=0}^n$ *be a martingale such that* $|X_i - X_{i-1}| \le c_i$ *for any* $1 \le i \le n$. *Then* $\mathsf{P}(|X_n - X_0| \ge x) \le 2e^{-\frac{x^2}{2\sum_{i=1}^n c_i^2}}$ *for any* $x > 0$.

In order to apply this inequality and prove Theorem 3, one can consider the martingale $X_i(d) = \mathsf{E}(N_n(d) \mid G_m^i)$, $i = 0, \dots, n$. Note that $X_0(d) = \mathsf{E}N_n(d)$ and $X_n(d) = N_n(d)$. It remains to estimate the difference $|X_i(d) - X_{i-1}(d)|$. The upper bound for $|X_i(d) - X_{i-1}(d)|$ can be proven by induction on d and n. The complete proofs of Theorems 2 and 3 can be found in [37].

Theorems 2 and 3 imply that the degree distribution in G_m^n follows (asymptotically) the power law with the parameter $1 + \frac{1}{A}$. Recall that in this case the *cumulative* degree distribution follows the power law with the parameter $\gamma = \frac{1}{A}$.

4 Clustering coefficient in PA-class

In the previous section, we described a general class of preferential attachment models and discussed the degree distribution in this class. The next natural question is: can we also say something about the clustering coefficient in this class? This question was discussed in [26,37].

Recall that there are two well-known definitions of the clustering coefficient of a graph G. The global clustering coefficient $C_1(G)$ is the ratio of three times the number of triangles to the number of pairs of adjacent edges in G. The average local clustering coefficient is $C_2(G) = \frac{1}{n}\sum_{i=1}^n C(i)$, where $C(i) = \frac{T^i}{P_2^i}$ is the local clustering coefficient for a vertex i, T^i is the number of edges between the neighbors of the vertex i, and P_2^i is the number of pairs of neighbors.

Some known results on the behavior of the clustering coefficient for classical preferential attachment models were mentioned in Sect. 2. In this section, we cover an approach which generalizes these results.

4.1 T-subclass

It turns out that models from the PA-class may have very different clustering coefficients even for fixed parameters A and m. Therefore, in order to be able to analyze the behavior of the clustering coefficients, we have to add some additional constraint.

In [37], a T-subclass of the PA-class was introduced. In order to belong to the T-subclass, a model has to satisfy the following property in addition to (3)–(6):

$$\mathbf{P}(d_i^{n+1} = d_i^n + 1, d_j^{n+1} = d_j^n + 1 \mid G_m^n) = e_{ij}\frac{D}{mn} + O\left(\frac{d_i^n d_j^n}{n^2}\right). \tag{7}$$

Here e_{ij} is the number of edges between vertices i and j in G_m^n and D is a non-negative constant. Note that this property still does not define the correlation between m edges completely.

All preferential attachment models described in Sect. 2 belong to the T-subclass: the LCD and the Buckley–Osthus models with $D = 0$, the Holme–Kim model with $D = P_t \cdot (m-1)$, and the RAN model with $D = 3$.

In Sects. 4.2 and 4.3, we discuss the behavior of global and average local clustering coefficients in the T-subclass. These results generalize several known ones independently obtained for the LCD, the Buckley–Osthus, the Holme–Kim, and the RAN models.

4.2 Global clustering coefficient

In this section, we discuss the global clustering coefficient for the models from the T-subclass. Recall that $C_1(G_m^n)$ is the ratio of three times the number of triangles to the number of pairs of adjacent edges in G_m^n. Therefore, we first study the random variable $P_2(n)$ which is equal to the number of P_2's (pairs of adjacent edges) in a random graph G_m^n.

Theorem 5 (Ostroumova et al. [37]) *For every model from the PA-class and for any $\varepsilon > 0$, we have*

(1) *if* $2A < 1$, *then* **whp** $(1-\varepsilon)\left(2m(A+B)+\frac{m(m-1)}{2}\right)\frac{n}{1-2A} \le P_2(n) \le (1+\varepsilon)\left(2m(A+B)+\frac{m(m-1)}{2}\right)\frac{n}{1-2A}$;

(2) *if* $2A = 1$, *then* **whp** $(1-\varepsilon)\left(2m(A+B)+\frac{m(m-1)}{2}\right)n\log(n) \le P_2(n) \le (1+\varepsilon)\left(2m(A+B)+\frac{m(m-1)}{2}\right)n\log(n)$;

(3) *if* $2A > 1$, *then* **whp** $n^{2A-\varepsilon} \le P_2(n) \le n^{2A+\varepsilon}$.

Note that Theorem 5 does not require the condition (7) to be satisfied. Also, it is worth noting that the value $P_2(n)$ in scale-free graphs is usually determined by the power-law exponent γ. Indeed, we have $P_2(n) = \sum_{d=1}^{d_{\max}} N_n(d)\frac{d(d-1)}{2} \propto \sum_{d=1}^{d_{\max}} nd^{1-\gamma}$, where $d_{\max}$ is the maximum degree of a vertex. Therefore, if $\gamma > 2$, then $P_2(n)$ is linear in n. However, if $\gamma \le 2$, then $P_2(n)$ is superlinear.

Next, we look at the random variable $T(n)$ which is equal to the number of triangles in G_m^n. Note that in any model from the PA-class we have $T(n) = O(n)$ since at each step we add at most $\frac{m(m-1)}{2}$ triangles. If we combine this fact with the previous observation for $P_2(n)$, we see that if $\gamma \le 2$, then in any preferential attachment model (in which out-degrees of vertices are bounded) the global clustering coefficient tends to zero as n grows.

Theorem 6 (Ostroumova et al. [37]) *Let G_m^n satisfy the condition* (7) *with $D > 0$. Then for any $\varepsilon > 0$* **whp** $(1-\varepsilon)\,Dn \le T(n) \le (1+\varepsilon)\,Dn$.

The proof of this theorem is straightforward. The expectation of the number of triangles we add at each step is $D + o(1)$ (see Eq. (7)). The fact that the sum of $O\left(\frac{d_i^n d_j^n}{n^2}\right)$ over all *adjacent* vertices is $o(1)$ can be shown by induction using the conditions (3)–(6). It is also possible to first prove that the maximum degree grows as n^A and then use this fact to estimate the sum of the error terms. Therefore, $\mathsf{E}T(n) = Dn + o(n)$. The Azuma–Hoeffding inequality can be used to prove concentration.

As a consequence of Theorems 5 and 6, we get the following result on the global clustering coefficient $C_1(G_m^n)$.

Theorem 7 (Ostroumova et al. [37]) *Let G_m^n belong to the T-subclass with $D > 0$. Fix any $\varepsilon > 0$, then*

(1) *If $2A < 1$, then* ***whp*** $\frac{6(1-2A)D-\varepsilon}{m(4(A+B)+m-1)} \le C_1(G_m^n) \le \frac{6(1-2A)D+\varepsilon}{m(4(A+B)+m-1)}$;

(2) *If $2A = 1$, then* ***whp*** $\frac{6D-\varepsilon}{m(4(A+B)+m-1)\log n} \le C_1(G_m^n) \le \frac{6D+\varepsilon}{m(4(A+B)+m-1)\log n}$;

(3) *If $2A > 1$, then* ***whp*** $n^{1-2A-\varepsilon} \le C_1(G_m^n) \le n^{1-2A+\varepsilon}$.

Note that in some cases ($2A \ge 1$, i.e., $\gamma \le 2$) the global clustering coefficient $C_1(G_m^n)$ tends to zero (for any D) as the number of vertices grows. A generalization of the obtained result to scale-free graphs will be discussed in Sect. 5.

In the next section, we look at the average local clustering coefficient and show that it behaves differently.

4.3 Average local clustering coefficient

In this section, we analyze the behavior of the average local clustering coefficient $C_2(G_m^n)$. First, we can easily show that $C_2(G_m^n)$ does not tend to zero if the condition (7) holds with $D > 0$. From Theorems 2 and 3 it follows that **whp** the number of vertices of degree m in G_m^n is greater than cn for some positive constant c. The expectation of the number of triangles we add at each step is $D + o(1)$. Therefore, **whp**

$$C_2(G_m^n) \ge \frac{1}{n} \sum_{i:\deg(i)=m} C(i) \ge \frac{2cD}{m(m+1)}.$$

So, if $D > 0$, then the local clustering coefficient does not tend to zero, as it is observed in real networks.

In [26] the local clustering coefficient was studied deeper. Namely, the authors analyze the function $C_2(d)$—the local clustering coefficient for the vertices of degree d—in the PA-class of models.

It was previously shown that in real-world networks $C_2(d)$ usually decreases as $d^{-\psi}$ with some parameter $\psi > 0$ [15,40,42]. For some networks, $C_2(d)$ scales as d^{-1} [27,39]. It turns out that in *all* models of the T-subclass the local clustering coefficient $C_2(d)$ asymptotically behaves as $\frac{2D}{Am} \cdot d^{-1}$.

Let $T_n(d)$ be the number of triangles on the vertices of degree d in G_m^n (i.e., the number of edges between the neighbors of the vertices of degree d). Then, the average local clustering coefficient for the vertices of degree d is defined as

$$C_2(d) = \frac{T_n(d)}{N_n(d)\frac{d(d-1)}{2}}. \quad (8)$$

In other words, $C_2(d)$ is the local clustering coefficient averaged over all vertices of degree d.

In Sect. 3.2, we already discussed the asymptotic behavior of $N_n(d)$. Therefore, it remains to estimate $T_n(d)$. The following theorem on the expectation of $T_n(d)$ is proven in [26].

Theorem 8 (Krot et al. [26]) *Let G_m^n belong to the T-subclass with $D > 0$. Then*

(1) *if* $2A < 1$, *then* $\mathsf{E}T_n(d) = K(d)(n + O(d^{2+\frac{1}{A}}))$;
(2) *if* $2A = 1$, *then* $\mathsf{E}T_n(d) = K(d)(n + O(d^{2+\frac{1}{A}} \cdot \log(n)))$;
(3) *if* $2A > 1$, *then* $\mathsf{E}T_n(d) = K(d)(n + O(d^{2+\frac{1}{A}} \cdot n^{2A-1}))$;

where $K(d) = c(m,d)\left(D + \frac{D}{m} \cdot \sum_{i=m}^{d-1} \frac{i}{Ai+B}\right) \overset{d\to\infty}{\sim} \frac{D}{Am} \cdot \frac{\Gamma\left(m+\frac{B+1}{A}\right)}{A\,\Gamma\left(m+\frac{B}{A}\right)} \cdot d^{-\frac{1}{A}}$.

In addition, $T_n(d)$ is highly concentrated around its expectation.

Theorem 9 (Krot et al. [26]) *Let G_m^n belong to the T-subclass with $D > 0$. Then for every $d = d(n)$*

(1) *if* $2A < 1$: $\mathsf{P}(|T_n(d) - \mathsf{E}T_n(d)| \geq d^2\sqrt{n}\log n) = O(n^{-\log n})$;
(2) *if* $2A = 1$: $\mathsf{P}(|T_n(d) - \mathsf{E}T_n(d)| \geq d^2\sqrt{n}\log^2 n) = O(n^{-\log n})$;
(3) *if* $2A > 1$: $\mathsf{P}(|T_n(d) - \mathsf{E}T_n(d)| \geq d^2 n^{2A-\frac{1}{2}}\log n) = O(n^{-\log n})$.

Consequently, for any $\delta > 0$ there exists a function $\varphi(n) = o(1)$ such that

(1) *if* $2A \leq 1$: $\lim_{n\to\infty} \mathsf{P}\left(\exists\, d \leq n^{\frac{A-\delta}{4A+2}} : |T_n(d) - \mathsf{E}T_n(d)| \geq \varphi(n)\,\mathsf{E}T_n(d)\right) = 0$;
(2) *if* $2A > 1$: $\lim_{n\to\infty} \mathsf{P}\left(\exists\, d \leq n^{\frac{A(3-4A)-\delta}{4A+2}} : |T_n(d) - \mathsf{E}T_n(d)| \geq \varphi(n)\,\mathsf{E}T_n(d)\right) = 0$.

As a corollary of Theorems 2, 3, 8, and 9, we get the following result on the average local clustering coefficient $C_2(d)$ for the vertices of degree d.

Theorem 10 (Krot et al. [26]) *Let G_m^n belong to the T-subclass of the PA-class. Then for any $\delta > 0$ there exists a function $\varphi(n) = o(1)$ such that*

(1) *if* $2A \leq 1$: $\lim_{n\to\infty} \mathsf{P}\left(\exists\, d \leq n^{\frac{A-\delta}{4A+2}} : \left|C_2(d) - \frac{K(d)}{\binom{d}{2}c(m,d)}\right| \geq \frac{\varphi(n)}{d}\right) = 0$;
(2) *if* $2A > 1$: $\lim_{n\to\infty} \mathsf{P}\left(\exists\, d \leq n^{\frac{A(3-4A)-\delta}{4A+2}} : \left|C_2(d) - \frac{K(d)}{\binom{d}{2}c(m,d)}\right| \geq \frac{\varphi(n)}{d}\right) = 0$.

Note that $\frac{K(d)}{\binom{d}{2}c(m,d)} = \frac{2D}{d\,(d-1)\,m}\left(m + \sum_{i=m}^{d-1}\frac{i}{Ai+B}\right) \overset{d\to\infty}{\sim} \frac{2D}{mA} \cdot d^{-1}$.

Finally, despite the fact that the T-subclass generalizes many different models, it is possible to analyze the local clustering coefficient for all these models. It turns out that $C_2(d)$ asymptotically decreases as $\frac{2D}{Am} \cdot d^{-1}$. In particular, this result implies that one cannot change the exponent -1 by varying the parameters A, D, and m. This basically means that preferential attachment models in general are not flexible enough to model $C(d) \propto d^{-\psi}$ with $\psi \neq 1$.

There is also a connection between the obtained result and the notion of *weak* and *strong transitivity* introduced in [40]. It was shown in [41] that percolation properties of a network are defined by the type (weak or strong) of its connectivity. Interestingly, a model from the T-subclass can belong to either weak or strong transitivity class: if $2D < Am$, then we obtain the weak transitivity; if $2D > Am$, then we obtain the strong transitivity.

5 Global clustering coefficient in scale-free networks

5.1 Motivation

While the degree distribution of preferential attachment models allows adjustment to reality, the clustering coefficient is difficult to model in some cases. For most real-world networks the parameter γ of their cumulative degree distribution belongs to the interval $(1, 2)$. As we showed in Sect. 4.2, once $\gamma < 2$ in a preferential attachment model, the global clustering coefficient decreases as the graph grows, which does not correspond to the majority of real-world networks [33]. The main reason of this decrease is that the number of edges added at each step is a constant and consequently the number of triangles can grow only linearly with the number of vertices n, while the number of pairs of adjacent edges grows as $n^{2/\gamma}$. Not only preferential attachment models suffer from this problem: to the best of our knowledge, in the literature there are no models of scale-free networks with an infinite variance of the degree distribution and with an asymptotically constant global clustering coefficient.

In this section, we address the above problem. In particular, we explain why such a model—one with a power-law degree distribution, with $\gamma < 2$, and with an asymptotically constant global clustering coefficient—cannot exist. In order to do this, we consider a sequence of graphs with degree distributions following a regularly varying distribution F. We assume that the degrees of the vertices are randomly generated according to F. Then, for a given outcome of the degree sequence, a graph can be built in any arbitrary way. We show that if a simple graph has a power-law degree distribution with an infinite variance, then the global clustering coefficient for any such sequence of graphs tends to zero with high probability. Note that we do not assume any random graph model here.

In addition to the upper bound obtained for the global clustering coefficient, we also present an algorithm which allows to construct graphs with nearly maximum (up to $n^{o(1)}$ multiplier) clustering coefficient for the considered sequence of graphs.

On the contrary, for weighted graphs, the constant global clustering coefficient can be obtained even for the case of an infinite variance of the degree distribution.

This section covers the results presented in [35,36].

5.2 Scale-free graphs

Let us consider a sequence of graphs $\{G_n\}$. Each graph G_n has n vertices. As in [35,36], we assume that the degrees of the vertices are independent random variables following a *regularly varying* distribution with a cumulative distribution function F satisfying

$$1 - F(x) = L(x)x^{-\gamma}, \quad x > 0, \tag{9}$$

where $L(\cdot)$ is a slowly varying function, that is, for any fixed constant $t > 0$

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1.$$

There is an additional obvious restriction on the function $L(\cdot)$: the function $1 - L(x)x^{-\gamma}$ must be a cumulative distribution function of a random variable taking positive integer values with probability 1.

Note that Eq. (9) generalizes the power-law distribution and describes a broad class of heavy-tailed distributions. Further by $\xi, \xi_1, \xi_2, \ldots$ we denote random variables with the distribution F. Note that for any $\alpha < \gamma$ the moment $\mathsf{E}\xi^\alpha$ is finite.

Models with $\gamma > 2$ and with the global clustering coefficient tending to some positive constant were already proposed (see, e.g., [37]). Therefore, further we consider only the most tricky case $1 < \gamma < 2$.

Obviously, we can construct a graph with a given degree distribution only if the sum of all degrees is even. This problem is easy to solve: we can either regenerate the degrees until their sum is even or we can add 1 to the last variable if their sum is odd [12]. As in [35], we choose the second option: if $\sum_{i=1}^n \xi_i$ is odd, then we replace ξ_n by $\xi_n + 1$. It is easy to see that this modification does not change any of obtained results, therefore, further we do not focus on the evenness.

Further in this section, we state that some results hold *with high probability*. Let us emphasize that the probability here only refers to the randomness defining the degree sequence, and the obtained bounds, e.g., $O(n^{-\alpha})$ with some $\alpha > 0$, hold uniformly with respect to any sequence of graphs $\{G_n\}$ with a given degree sequence.

5.3 Useful auxiliary results

In this section, we formulate several auxiliary theorems and lemmas which are used in the rest of the paper.

The following theorem can be very useful when one deals with the regularly varying distributions.

Theorem 11 (Karamata [6]) *Let L be slowly varying and locally bounded in* $[x_0, \infty]$ *for some* $x_0 \ge 0$. *Then*

(1) *for* $\alpha > -1$

$$\int_{x_0}^x t^\alpha L(t)dt = (1 + o(1))(\alpha + 1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty;$$

(2) *for* $\alpha < -1$

$$\int_x^\infty t^\alpha L(t)dt = -(1 + o(1))(\alpha + 1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty.$$

We need the following notation:

$$S_{n,c}(x) = \sum_{i=1}^n \xi_i^c I\,[\xi_i > x]\,,$$

$$\bar{S}_{n,c}(x) = \sum_{i=1}^n \xi_i^c I\,[\xi_i \le x]\,,$$

here $c, x \geq 0$.

Karamata's theorem allows to analyze the asymptotic behavior for $S_{n,c}(x)$ and $\bar{S}_{n,c}(x)$. Namely, the following lemmas hold.

Lemma 1 (Ostroumova Prokhorenkova [36]) *Fix any c such that $0 \leq c < \gamma$, any β such that $1 < \beta < \gamma/c$ and $\beta \leq 2$, and any $\varepsilon > 0$. Then for any $x = x(n) > 0$ such that $x(n) \to \infty$ we have*

$$\mathsf{E}S_{n,c}(x) = \frac{\gamma}{\gamma - c} n\, x^{c-\gamma} L\,(x)\,(1 + o(1)), \quad n \to \infty,$$

$$\mathsf{P}\left(|S_{n,c}(x) - \mathsf{E}S_{n,c}(x)| > \varepsilon\, \mathsf{E}S_{n,c}(x)\right) = O\left(\left(\frac{x^{\gamma}}{n\,L\,(x)}\right)^{\beta-1}\right).$$

Further in this paper we refer to this lemma only with $c = 0$. In this case we can take $\beta = 2$, since $\gamma/c = \infty$. Lemma 1 with $c = 1$ is needed to overcome some technical difficulties which are omitted in this paper (see [36] for the full proofs). If $c = 1$, then we have to chose some β such that $1 < \beta < \gamma$.

Lemma 2 (Ostroumova Prokhorenkova [36]) *Fix any c such that $c > \gamma$ and any $\varepsilon > 0$. Then for any $x = x(n) > 0$ such that $x(n) \to \infty$ we have*

$$\mathsf{E}\bar{S}_{n,c}(x) = \frac{\gamma}{c - \gamma} n\, x^{c-\gamma} L\,(x)\,(1 + o(1)), \quad n \to \infty,$$

$$\mathsf{P}\left(|\bar{S}_{n,c}(x) - \mathsf{E}\bar{S}_{n,c}(x)| > \varepsilon\, \mathsf{E}\bar{S}_{n,c}(x)\right) = O\left(\frac{x^{\gamma}}{n\,L\,(x)}\right).$$

We need two more lemmas. Put $\xi_{max} = \max\{\xi_1, \ldots, \xi_n\}$.

Lemma 3 (Ostroumova Prokhorenkova [36]) *For any $\varepsilon > 0$ and any $\alpha > 0$*

$$\mathsf{P}\left(\xi_{max} > n^{\frac{1}{\gamma}-\varepsilon}\right) = 1 - O(n^{-\alpha}).$$

Also, for any $\delta < \gamma\varepsilon$

$$\mathsf{P}\left(\xi_{max} \leq n^{\frac{1}{\gamma}+\varepsilon}\right) = 1 - O(n^{-\delta}).$$

Lemma 4 (Ostroumova Prokhorenkova [36]) *For any $\varepsilon > 0$ and any $\delta < \frac{\gamma\varepsilon}{\gamma+2}$*

$$\mathsf{P}\left(\bar{S}_{n,2}(\infty) \leq n^{\frac{2}{\gamma}+\varepsilon}\right) = 1 - O(n^{-\delta}).$$

Above we present only the upper bound for $\bar{S}_{n,2}(\infty)$, since the lower bound can be obtained using the lower bound for ξ_{max}: $\bar{S}_{n,2}(\infty) \geq \xi_{max}^2$.

5.4 Clustering in unweighted graphs

5.4.1 Existence

The behavior of the global clustering coefficient in scale-free unweighted (simple) graphs was considered in [35,36]. In the case of an infinite variance, the reasonable question is whether there exists a simple graph (i.e., a graph without loops and multiple edges) with a given degree distribution. As pointed out in [28], the probability of obtaining a simple graph with a given degree distribution by random pairing of edges' endpoints (configuration model) converges to a strictly positive constant only if the degree distribution has a finite second moment. In other words, if the second moment is infinite, then a random graph with a given degree distribution has loops or multiple edges asymptotically almost surely. However, the following theorem holds.

Theorem 12 (Ostroumova Prokhorenkova and Samosvat [35]) *For any δ such that $1 < \delta < \gamma$ with probability $1 - O\left(n^{1-\delta}\right)$ there exists a simple graph on n vertices with the degree distribution defined in Sect.* 5.2.

In order to prove this theorem, one can use Erdős–Gallai theorem.

Theorem 13 (Erdős and Gallai [20]) *A sequence of non-negative integers $d_1 \geq \ldots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if*

(1) $d_1 + \cdots + d_n$ *is even;*
(2) $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ *holds for* $1 \leq k \leq n$.

Let us order the realized values of the random variables $\xi_1, \ldots, \xi_n$ and obtain the ordered sequence $d_1 \geq \cdots \geq d_n$. Using the lemmas from Sect. 5.3, it is not hard to check that the conditions of Erdős–Gallai theorem hold for this sequence with probability $1 - O(n^{1-\delta})$. Therefore, Theorem 12 holds.

So, with high probability a graph G_n with the required degree distribution exists and it is reasonable to discuss its global clustering coefficient.

5.4.2 Upper bound

In this section, we discuss the following surprising fact: with high probability the global clustering coefficient for a sequence of graphs discussed in Sect. 5.2 tends to zero. Since Sect. 5.2 basically describes all graphs with an infinite variance of the power-law degree distribution, the obtained result contradicts a common belief that for many real-world networks the global clustering coefficients tends to a non-zero limit as the networks become large.

Theorem 14 (Ostroumova Prokhorenkova [36]) *For any $\varepsilon > 0$ and any α such that $0 < \alpha < \frac{1}{\gamma+1}$ with probability $1 - O(n^{-\alpha})$ the global clustering coefficient of G_n satisfies the following inequality*

$$C_1(G_n) \leq n^{-\frac{(2-\gamma)}{\gamma(\gamma+1)}+\varepsilon}.$$

Given the auxiliary results from Sect. 5.3, the proof of Theorem 14 is quite simple. Therefore, to provide a complete picture of the problem, we repeat this proof here.

Proof By the definition, the global clustering coefficient is $C_1(G_n) = \frac{3 \cdot T(n)}{P_2(n)}$, where $T(n)$ is the number of triangles and $P_2(n)$ is the number of pairs of adjacent edges in G_n.

Since $P_2(n) \geq \xi_{max}(\xi_{max} - 1)/2$, from Lemma 3 we get that for any $\delta > 0$ with probability $1 - O(n^{-\alpha})$

$$P_2(n) > n^{\frac{2}{\gamma} - \delta}.$$

It remains to estimate $T(n)$. The following observation is crucial for the proof. For any x

$$T(n) \leq |\{i : \xi_i > x\}|^3 + \sum_{i : \xi_i \leq x} \xi_i^2. \tag{10}$$

The first term in (10) is the upper bound for the number of triangles with all vertices among the set $\{i : \xi_i > x\}$. The second term is the upper bound for the number of triangles with at least one vertex among $\{i : \xi_i \leq x\}$.

From Lemmas 1 and 2 we get

$$|\{i : \xi_i > x\}| = S_{n,0}(x) \leq (1 + \varepsilon) n\, x^{-\gamma} L(x),$$
$$\sum_{i:\, \xi_i \leq x} \xi_i^2 = \bar{S}_{n,2}(x) \leq (1 + \varepsilon) \frac{\gamma}{2 - \gamma} n\, x^{2-\gamma} L(x)$$

with probability $1 - O\left(\frac{x^\gamma}{n\, L(x)}\right)$.

Now we can fix $x = n^{\frac{1}{\gamma+1}}$. So, with probability

$$1 - O\left(\frac{n^{-\frac{1}{\gamma+1}}}{L\left(n^{\frac{1}{\gamma+1}}\right)}\right) = 1 - O(n^{-\alpha})$$

we have

$$T(n) \leq n^{\frac{3}{\gamma+1} + \delta}.$$

Taking small enough δ, we obtain

$$C_1(G_n) \leq n^{\varepsilon - \frac{2-\gamma}{\gamma(\gamma+1)}}.$$

This concludes the proof. □

The obtained result is especially interesting due to the fact that in many observed networks the values of both clustering coefficients are considerably high [33]. Note that actually the observations from [33] do not contradict Theorem 14. There are several possible explanations:

- Large values of global clustering coefficient are usually obtained on small networks.
- For the networks with the power-law degree distribution the observed global clustering is usually less than the average local clustering, as expected.
- Our results can be applied only to networks with regularly varying degree distributions. If a network has, for example, a power-law degree distribution with an exponential cut-off, then our results cannot be applied.

5.4.3 Lower bound

In the previous section, we presented an upper bound for $C_1(G_n)$. It is reasonable to discuss the tightness of this upper bound.

Theorem 15 (Ostroumova Prokhorenkova [36]) *For any* $\varepsilon > 0$ *and any* α *such that* $0 < \alpha < min\{\frac{\gamma\varepsilon}{\gamma+2}, \frac{1}{\gamma+1}, \gamma - 1\}$ *with probability* $1 - O(n^{-\alpha})$ *there exists a graph with the degree distribution defined in Sect. 5.2 and the global clustering coefficient satisfying the following inequality*

$$C_1(G_n) \geq n^{-\frac{(2-\gamma)}{\gamma(\gamma+1)}-\varepsilon}.$$

Theorem 15 implies that the upper bound obtained in Theorem 14 is tight up to $n^{o(1)}$ multiplier. In other words, we know how many triangles we can construct if we fix a power-law degree distribution with $1 < \gamma < 2$.

The proof of Theorem 15 is quite simple. Here we present only the main idea and omit technical details. Recall that $C_1(G_n) = \frac{3 \cdot T(n)}{P_2(n)}$. The value $P_2(n)$ depends only on the degree sequence and the upper bound for $P_2(n)$ follows from Lemma 4 (this bound is of order $n^{\frac{2}{\gamma}}$). Therefore, it remains to construct a graph with a large number of triangles. It turns out that we can get enough triangles by constructing the largest possible clique in our graph. Indeed, it is easy to show that $n^{\frac{1}{\gamma+1}}$-th largest degree is of order $n^{\frac{1}{\gamma+1}}$. Therefore, the number of vertices in the largest possible clique is of order $n^{\frac{1}{\gamma+1}}$ and it gives about $n^{\frac{3}{\gamma+1}}$ triangles. Combining this with the upper bound for $P_2(n)$, we get the statement of the theorem.

5.5 Clustering in weighted graphs

In the previous section, we showed that with high probability the global clustering coefficient for a sequence of simple graphs defined in Sect. 5.2 tends to zero. However, it is also reasonable to study the global clustering coefficient for graphs with multiple edges. This agrees well with reality: for example, the Web host graph has a lot of multiple edges, since there can be several edges between the pages of two hosts. Even in the Internet graph (vertices are web pages and edges are links between them) multiple edges occur frequently.

There are several possible ways to define the global clustering coefficient for a weighted graph. The following reasonable generalization of the global clustering coefficient to multigraphs is proposed in [34]:

$$C_1(G) = \frac{\text{total value of closed triplets}}{\text{total value of triplets}}.$$

A triplet is a group of three vertices u, v, w such that the pairs u, v and u, w are connected. A triplet is called closed if v and w are also connected. Note that every triangle consists of three closed triplets. There are several ways to define the value of a triplet. First, the triplet value can be defined as the *arithmetic mean* of the weights of two edges (u, v) and (u, w) that make up the triplet. Second, it can be defined as the *geometric mean* of the weights of the edges. Third, it can be defined as the *maximum or minimum value* of the weights of the edges. In addition to these methods proposed in [34], the following natural definition of the weight is proposed in [36]: the weight of a triplet is the *product* of the weights of the edges. This definition agrees with the following property: the total value of all triplets located on a vertex is close to its degree squared. In the case of multigraphs, the weight of an edge is equal to its multiplicity.

Further we assume that loops are not allowed. It can be proven that even with this restriction it is possible to obtain an asymptotically constant global clustering coefficient. The following theorem holds for *any* definition of the global clustering coefficient $C_1(G_n)$.

Theorem 16 (Ostroumova Prokhorenkova [36]) *Fix any* $\delta > 0$. *For any* α *such that* $0 < \alpha < \frac{\gamma-1}{\gamma+1}$ *with probability* $1 - O\left(n^{-\alpha}\right)$ *there exists a loopless multigraph with the degree distribution defined in Sect.* 5.2 *and the global clustering coefficient satisfying the following inequality*

$$C_1(G_n) \geq \frac{2-\gamma}{2+\gamma} - \delta.$$

The formal proof of this theorem can be found in [36]. The main idea is the following. Up to a slowly varying multiplier, the size of the largest possible clique is $n^{\frac{1}{\gamma+1}}$. As we already discussed, this clique can be constructed on the vertices of largest degrees. Since we want to get as many closed triplets as possible, we not only construct a clique on the vertices of largest degrees, but we construct a *multiclique*, i.e., we want these $n^{\frac{1}{\gamma+1}}$ vertices to be connected only to each other. This gives us a lower bound $\frac{1}{2}\, n^{\frac{3}{\gamma+1}}$ (up to a slowly varying multiplier) for the total value of closed triplets. Now it remains to get an upper bound for the total value of all triplets. This value consists of the total value of closed triplets estimated above plus the total value of triplets located on the vertices of small (less than $n^{\frac{1}{\gamma+1}}$) degrees. The latter value can be estimated by the sum of squares of the degrees of these vertices, which is of order $\frac{\gamma}{2-\gamma}\, n^{\frac{3}{\gamma+1}}$. Altogether, this leads to the statement of Theorem 16. Note that all the above estimates hold for any definition of the weight of a triplet.

6 Conclusion

In this paper, we reviewed several recent results on preferential attachment models and clustering coefficient analysis. The main aim was to cover *general approaches* to network analysis. It turns out that in some cases it is not necessary to fully specify a random graph model in order to study its properties.

In particular, we presented a general class of preferential attachment models (PA-class) defined in terms of constraints that are sufficient for the study of the degree distribution. Then, we discussed the T-subclass, where an additional constraint is added in order to be able to analyze the clustering coefficient. Finally, in Sect. 5, we discussed the global clustering coefficient under only one constraint: we specified only the degree distribution.

We believe that this paper will motivate further general studies of complex networks and their properties.

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